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APPROXIMATIONS OF FUNCTIONS BY SETS OF POLES

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FOREWORD

The choice of functions to be approximated for this report was determined by the requirements of a project for the computation of surface waves in the wake of a ship.

The computation of rational functions was programmed originally for the Naval Ordnance Research Calculator. The development of rational functions was pushed to a sixteen decimal digit accuracy before the destruction of NORC would make all this programming useless for further computation. A sixteen decimal digit level of accuracy was selected because this is the level for double precision computation on the IBM 360 computers. It is more accurate than necessary for single precision computation on the CDC 6600 computers, but no attempt will be made to readjust the approximations back to a lower level of accuracy.

Computation of constants and programming of checkouts were contributed by Mrs. E. J. Hershey and by Mr. W. H. Langdon. The manuscript was completed by 11 July, 1971. The report was reviewed administratively by Mr. J. H. Walker, Jr.

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ABSTRACT

A few special functions have been approximated in the complex plane with rational functions. The error bounds of the approximations conform to the Chebyshev criterion. The rational functions have been converted into equivalent expansions in terms of the singular functions for poles and residues. Analyses and programs are described for ten functions.

INTRODUCTION

The choice of method which will be used in the preparation of a special–purpose subroutine is dictated by the requirement for efficiency in the specific application for which the subroutine will be used. The choice of method which will be used in the preparation of a general–purpose subroutine is dictated by the requirement for versatility in a computer system. A sensible policy is to prepare a subroutine in such a way that the rounding error in output is compatible with the inherent error in input for any single precision computation within the capacity of the computer. With such a policy it is likely that the subroutine will have the greatest range of application. The effort which has been expended in its preparation will have the greatest reward.

A number of special functions can be expressed as the sum of an ascending power series in the argument or as the sum of a descending power series in the argument.

The ascending power series is absolutely convergent. The truncation error in a finite sum can be made less than any arbitrary bound by taking enough terms in the series. When the argument is outside a circle of unit radius the magnitudes of the terms may increase at first with increasing order to a maximum magnitude although they always decrease ultimately with increasing order. When each term is computed to a finite number of digits the rounding error in the sum is determined principally by the rounding error in the largest term. In that part of the complex plane where the terms all have the same sign the rounding error is relatively ineffective because the sum is larger than the largest term, but in that part of the complex plane where the terms have alternating signs, the rounding error can be completely fatal because the sum is very much smaller than the largest term. The ascending power series is inadequate outside this contour of limited extent because of rounding error.

The descending power series is asymptotic. The magnitudes of the terms decrease or increase at first with increasing order, but they always decrease to a minimum, and finally increase with increasing order. Where the terms alternate in sign, the truncation error in a finite sum is less than the smallest term included. The truncation error is less than any arbitrary bound when the argument is outside a circle of sufficiently large radius. The descending power series is inadequate inside this circle of large radius.

There is a zone between the circle of unit radius and the circle of large radius which cannot be reached by either of the classical series. The conventional approach to this region has been to seek a converging factor which can be applied either to the leading term or to the smallest term of the asymptotic series and thereby achieve a satisfactory level of accuracy. Various schemes have been proposed in the literature for evaluating the converging factor.

In the method of Airey[1], the smallest term is factored out of the following divergent terms to obtain a series of divergent factors, each of which is expanded then in inverse powers of the order. Successive terms with the same inverse power of the order are replaced by their Euler sums. The terms in the summation converge for a limited range of the argument, but the results of summation are rational functions which can be evaluated at any argument.

Although the substitution of an Euler sum for a divergent series is dubious, the Airey formulae for the exponential integral and the probability integral have been confirmed rigorously by Murnaghan and Wrench[5,6] through transformations of variable and integrations by parts. They have extended the Airey converging factors to many more terms. Although their formulae give a great improvement in accuracy when the smallest asymptotic term is small anyway, they are not especially helpful when the

argument is near the unit circle. It is not clear that the terms in the converging factor are not themselves asymptotic. Cf. Page 22 of Reference 5.

In the method of Stieltjes[2], the integral representations of the Bessel functions are converted into double integrals whose integrands contain a binomial in the denominator. The reciprocal of the binomial is expanded in power series with remainder. Integration of the power series term by term recovers the terms of the asymptotic series while the remainder is retained as an integral. A transformation of the variables of integration and an expansion of the integrand in a power series provides an evaluation of the remainder in inverse powers of the order. This method is on firm ground insofar as its errors are limited to the approximation of an otherwise exact formula for the remainder, but it requires the evaluation of an exponential function in the evaluation of the remainder.

The Stieltjes method has been extended to more functions and to higher orders by J. T. Moore[3,4] in this laboratory. Insofar as the Airey method and the Stieltjes method both approximate the remainder in a series of descending powers of the order, it would be expected that the methods would be equivalent, and this has been confirmed by comparisons in Reference 27.

In the method of Dingle[9,10], basic functions are defined by the Borel sums of their asymptotic expansions. The remainders of the asymptotic expansions provide a set of basic converging factors. In the asymptotic expansion of the confluent hypergeometric function the remainder is an integral. The converging factor of the hypergeometric function is expressed as an integral of the basic converging factors. The integrand of the converging factor is expanded as a series in ascending powers of the argument and is integrated term by term. The method is justified insofar as it reconstructs the integral representation of the confluent hypergeometric function. Dingle's method has been determined, at least in the case of Bessel functions, to be itself asymptotic with significant truncation and rounding errors. Cf. Reference 27.

In a study by Tonelli[11], consideration has been given to the uniqueness and convergence of polynomial approximations which conform to the Chebyshev criterion on a closed contour in the complex plane.

In the method of Hastings[12], the converging factor would be approximated by a rational function with an error which conforms to the Chebyshev criterion. The maximum absolute error in each loop of the error curve would be the same in every loop.

The absolute value of any analytic function within a closed contour is everywhere less than its absolute value on the closed contour. The error is analytic if the function and its approximation both are analytic. The rational approximation is analytic if the roots of its denominator lie outside of the contour. The error then is bounded everywhere within the contour if it is bounded on the contour. If the contour lies within the region which can be reached by one of the classical series, but encloses the critical region which cannot be reached by either of the classical series, then the approximation will span the critical region.

The rational approximation can be specified at a series of points along the contour[13]. The points of specification can be situated at antinodal points or at nodal points in the variation of error along the contour. The antinodal specification would give a lower error bound, but it would be difficult to control the number and location of the antinodal points. In the nodal specification there is a maximum of error between each consecutive pair of nodes. The nodal points can be adjusted so as to equalize the maxima of error between nodes. The algorithm which was used in the development of rational approximations is given in Appendix A.

An effective contour of approximation lies along a half circle of unit radius, along the imaginary axis, and along a half circle of infinite radius to form a closed contour which contains the critical region of approximation. Although the error of approximation is bounded only within the contour of approximation, the rational approximation is good enough to be used in an extensive region outside of the contour of approximation. The rational approximation and the convergent series together can span the entire complex plane with little more error than the inherent error which arises from uncertainty in the argument itself. The boundary line between the regions of application of the different formulae is determined empirically.

The preparation of rational approximations with single precision requires source data with higher precision to serve as foundation and standard. The achievement of better than single precision accuracy was not possible in the case of Bessel functions without recourse to converging factors. The achievement of better than single precision accuracy was possible in the case of other functions through sixteen–point Gaussian integration with double precision. The numerical constants for the double precision computations are available in the literature[16–20].

The rational approximations can be expanded into the sums of singular functions which are useful for further analysis. The singular functions represent poles and residues at the roots of the denominators of the rational approximations. An algorithm for the conversion of a rational function into a polar expansion is given in Appendix A. The results of approximation are given in Appendix B.

The errors of approximation are located on spiral curves in the complex plane. Representative curves are illustrated in References 25 and 26. Maximum errors in the polar approximations along the imaginary axis and along the arc of unit radius are tabulated in Appendix C.

Justification for the success of the polar expansion is to be found in a paper by Gautschi[14] on the error function. The asymptotic expansion of the error function is identified with the asymptotic expansion of a Stieltjes integral, and approximation of the Stieltjes integral by Gauss–Hermite quadrature leads to an expansion in poles and residues.

In the method of Goldstein and Thaler[15], the Bessel functions are evaluated by a cycling of recurrence equations from such a large order that the Bessel function is negligible in comparison with the Weber function. A large number of cycles of recurrence is required when the argument is large.

CUTS

For a general purpose subroutine which generates a complex function of a complex argument it is desirable to restrict the range of argument by suitable cuts in the complex plane. Otherwise the function to be generated would be multiple valued as the argument encircles a singularity. The call lines of the subroutine would be encumbered with parameters which would specify the number of times that the argument has encircled each singularity. In the present system, it is assumed that the complex plane is cut by straight lines which extend from each singularity to infinite distance in a direction radially outward from the origin. Whenever the argument happens to fall exactly on a cut, it is assumed to belong to that side from which it would come during a counterclockwise displacement around the singularity. When the function to be computed is for a path which crosses the cut, then the path can be varied to encircle the singularity and the function can be incremented by its change of value during encirclement.

COMPLEX GAMMA FUNCTION

Analysis

The gamma function $\Gamma(z)$ is defined by the equation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left[\frac{e^{\frac{z}{n}}}{1 + \frac{z}{n}} \right] \tag{1}$$

where γ is Euler's constant. The gamma function has poles at the negative integers such that the residue of the nth pole is $(-1)^n/n!$.

For a small argument the reciprocal of the gamma function is given by the Bourguet convergent series and for a large argument the logarithm of the gamma function is given by the Stirling asymptotic series. Intermediate regions can be spanned by recurrence relations. A rational approximation is not necessary.

The gamma function of an argument with a negative real part is expressed in terms of the gamma function of an argument with a positive real part by the reciprocal equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{2}$$

It is necessary to evaluate series expansions only for arguments with positive real parts.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{3}$$

then the gamma function is derived from an ascending power series. The reciprocal of the gamma function is given by the equation

$$\frac{1}{\Gamma(1+z)} = \sum_{m=0}^{\infty} c_m z^m \tag{4}$$

for which the coefficients c_m are listed on page 256 of Reference 21.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 32 \tag{5}$$

then the gamma function is computed from a descending power series. From the equations on page 252 of Reference 23, the logarithm of the gamma function is given by the equation

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log (2\pi) + \sum_{m=1}^{N-1} \frac{(-1)^{m-1} B_m}{2m(2m-1) z^{2m-1}} \tag{6}$$

where the Bernoulli numbers B_m are defined by the equation

$$\frac{B_m}{(2m)!} = \frac{2}{(2^{2m}-1)\pi^{2m}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m}} \tag{7}$$

Summation of the series is continued until there is no change in sum or until $m = 18$.

If the argument $x + iy$ satisfies the inequality

$$1 < x^2 + y^2 < 32 \tag{8}$$

then the gamma function is computed with the aid of the difference equation

$$\Gamma(1 + z) = z\Gamma(z) \tag{9}$$

If n is the integer which is nearest in value to x and if n satisfies the inequality

$$|z - n|^2 \leqq 1 \tag{10}$$

then the gamma function is given by the equation

$$\Gamma(z) = (z - 1)\cdots(z - n + 1)\Gamma(z - n + 1) \tag{11}$$

for which $\Gamma(z - n + 1)$ is evaluated from the convergent series. If n is the smallest integer which satisfies the inequality

$$|z + n|^2 \geqq 32 \tag{12}$$

then the gamma function is given by the equation

$$\Gamma(z) = \frac{\Gamma(z + n)}{z\cdots(z + n - 1)} \tag{13}$$

for which $\Gamma(z + n)$ is evaluated from the asymptotic series.

Programming

SUBROUTINE CGAMMA (MO, AZ, FG)

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FORTRAN SUBROUTINE FOR COMPLEX GAMMA FUNCTION
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The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The complex gamma function is computed by series expansions and recurrence relations. If MO = 0, the real and imaginary parts of the function $\Gamma(z)$ are stored in array FG. If MO = 1, the real and imaginary parts of the function $\log \Gamma(z)$ are stored in array FG.

COMPLEX LOGARITHMIC INTEGRAL

Analysis

The logarithmic integral $L(1+z)$ is defined by the equation

$$L(1+z) = \int_0^z \frac{1}{t} \log(1+t)\, dt \tag{14}$$

The integrand is analytic at $t=0$ but has a singularity at $t=-1$. The complex plane is cut from the singularity to $-\infty$ along the negative real axis.

The entire complex plane is covered by a pair of series which are centered at the origin. The rates of convergence of the series are slow in an annular zone which contains their common circle of convergence. The annular zone is covered by a pair of series which are centered at the singularity. A rational approximation is not necessary.

If the argument $x+iy$ satisfies the inequality

$$x^2 + y^2 \leqq \tfrac{1}{4} \tag{15}$$

then a Taylor series expansion of the logarithm in the integrand leads to the series

$$L(1+z) = -\sum_{m=0}^{\infty} \frac{(-z)^{m+1}}{(m+1)^2} \tag{16}$$

which is convergent for z inside the circle $|z|=1$.

If the argument $x+iy$ satisfies the inequality

$$x^2 + y^2 \geqq 9 \tag{17}$$

then the logarithm in the integrand is replaced in accordance with the equation

$$\log(1+t) = \log t + \log\left(1 + \frac{1}{t}\right) \tag{18}$$

Expansion in powers of $1/t$ leads to the equation

$$L(1+z) = \frac{\pi^2}{6} + \sum_{m=0}^{\infty} \frac{1}{(m+1)^2(-z)^{m+1}} + \tfrac{1}{2}\log^2 z \tag{19}$$

in which the constant of integration is derived from the equation

$$\int_0^1 \frac{1}{t}\log(1+t)\, dt = \frac{\pi^2}{12} \tag{20}$$

and the series is convergent for z outside the circle $|z|=1$.

If the argument $x+iy$ satisfies the inequality

$$(1+x)^2 + y^2 \leqq \tfrac{1}{4} \tag{21}$$

then integration by parts gives the equation

$$\int_0^z \frac{1}{t}\log(1+t)\, dt = \log(-z)\log(1+z) - \int_0^z \frac{\log(-t)}{1+t}\, dt \tag{22}$$

Expansion in powers of $1+t$ leads to the equation

$$L(1+z) = -\frac{\pi^2}{6} + \sum_{m=0}^{\infty} \frac{(1+z)^{m+1}}{(m+1)^2} + \log(-z)\log(1+z) \tag{23}$$

in which the constant of integration is derived from the equation

$$\int_0^{-1} \frac{1}{t}\log(1+t)\,dt = -\frac{\pi^2}{6} \tag{24}$$

and the series is convergent for z inside a circle with center at -1 and with unit radius.

If the argument $x+iy$ satisfies none of the inequalities above, then let z be replaced by u in accordance with the equation

$$u = -\log(1+z) \tag{25}$$

whence the integral is transformed in accordance with the equation

$$\int_0^{z} \frac{1}{t}\log(1+t)\,dt = -\int_0^{u} \frac{t}{e^t-1}\,dt \tag{26}$$

From formula 764, page 90 of Peirce's *Table of Integrals*[22] the integral is given by the equation

$$L(1+z) = -u + \frac{u^2}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{(2n+1)!}u^{2n+1} \tag{27}$$

where the B_n are Bernoulli numbers. The series converges when z lies within a pair of contours which are concentric with the singularity at -1. The contours are defined by the equation $|u| = 2\pi$ and enclose all of the other circles of convergence.

Programming

SUBROUTINE CLGMCI (AZ, FL)

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FORTRAN SUBROUTINE FOR COMPLEX LOGARITHMIC INTEGRAL
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The real and imaginary parts of the argument z are given in array AZ. The complex logarithmic integral is computed by Taylor and Bernoulli expansions. The real and imaginary parts of the function $L(1+z)$ are stored in array FL.

COMPLEX EXPONENTIAL INTEGRAL

Analysis

The complex exponential integral $Ei(z)$ can be defined in terms of its complex argument z by the equation

$$Ei(z) = \int_{-\infty}^{z} \frac{e^t}{t}\, dt \tag{28}$$

where the path of integration lies in that part of the complex plane from which the positive real axis is excluded. The conventional exponential integral of real argument x is the real part of the complex integral when the complex argument z is real. Substitution of $-t$ for t gives the equation

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt = \mathcal{R}e Ei(x \pm 0i) \tag{29}$$

The sine integral $Si(x)$ is defined by the equation

$$Si(x) = \int_{0}^{x} \frac{\sin t}{t}\, dt \tag{30}$$

and the cosine integral $Ci(x)$ is defined by the equation

$$Ci(x) = -\int_{|x|}^{\infty} \frac{\cos t}{t}\, dt \tag{31}$$

The conventional integrals are recovered from the complex integral by the transformations

$$t \to it \qquad\qquad z \to ix \tag{32}$$

and are expressed by the equation

$$Ci(x) + iSi(x) = \pm\, i\frac{\pi}{2} + Ei(ix) \tag{33}$$

where the sign of $\pm\, i\frac{\pi}{2}$ is the same as the sign of x.

The Stieltjes form of the exponential integral is given by the equation

$$Ei(z) = e^z \int_{0}^{\infty} \frac{e^{-u}}{z-u}\, du \tag{34}$$

which may be derived from the substitution $t = z - u$.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{35}$$

or both of the inequalities

$$x^2 + y^2 < 1600 \qquad\qquad -x + 0.064\, y^2 \leqq 0 \tag{36}$$

then the exponential integral is computed from the ascending power series. Substitution

of $-t$ for t in the complex integral leads to the equation

$$Ei(z) = \int_0^1 \frac{1 - e^{-t}}{t}\, dt - \int_1^\infty \frac{e^{-t}}{t}\, dt + \int_1^{-z} \frac{dt}{t} - \int_0^{-z} \frac{1 - e^{-t}}{t}\, dt \tag{37}$$

Evaluation of integrals and expansion in series leads to the classical equation

$$Ei(z) = \gamma + \log(-z) + \sum_{m=1}^{\infty} \frac{z^m}{m \cdot m!} \tag{38}$$

for which the constant of integration is Euler's constant γ and is defined by the equation

$$\gamma = \int_0^1 \frac{1 - e^{-t} - e^{-\frac{1}{t}}}{t}\, dt \tag{39}$$

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1600 \qquad\qquad -x + 0.064\, y^2 > 0 \tag{40}$$

then the exponential integral is computed from the rational approximation. When the converging factor is approximated by poles and residues, then the exponential integral is expressed by the equation

$$Ei(z) = e^z \sum_{i=1}^{18} \frac{\epsilon_i}{z - \delta_i} \tag{41}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table I.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1600 \tag{42}$$

then the exponential integral is computed from the descending power series. Repeated integration by parts leads to the equation

$$Ei(z) = \frac{e^z}{z} \sum_{m=0}^{N-1} \frac{m!}{z^m} + N! \int_{-\infty}^{z} \frac{e^t}{t^{N+1}}\, dt \tag{43}$$

in which the series is asymptotic. Summation of the series is continued until there is no change in the sum or until $m = 39$.

Programming

SUBROUTINE CEXPLI (MO, AZ, FE)

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FORTRAN SUBROUTINE FOR COMPLEX EXPONENTIAL INTEGRAL

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The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The complex exponential integral is computed by series expansions and rational approximations. If MO = 0, the real and imaginary parts of the function $Ei(z)$ are stored in array FE. If MO = 1, the real and imaginary parts of the function $e^{-z}Ei(z)$ are stored in array FE.

COMPLEX FRESNEL INTEGRAL

Analysis

Various forms of complex integral with different constants of integration can be utilized. Let the complex Fresnel integral $E(z)$ for complex argument z be defined by the equation

$$E(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}}\, dt \tag{44}$$

where the path of integration lies within that part of the complex plane from which the positive real axis is excluded. The phase of z is limited to the range 0 to 2π, and the phase of $z^{1/2}$ is half the phase of z.

The conventional Fresnel integrals are defined in terms of harmonic functions by the equations

$$C(v) + iS(v) = \int_0^v e^{\frac{1}{2}\pi i u^2}\, du = \int_0^v i^{u^2}\, du \tag{45}$$

or are expressed in terms of Hankel functions by the equations

$$C(x) + iS(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{e^{iu}}{u^{\frac{1}{2}}}\, du = \frac{i}{2} \int_0^x H_{\frac{1}{2}}^{(1)}(u)\, du \tag{46}$$

where x and v are related by the equation

$$x = \tfrac{1}{2}\pi v^2 \tag{47}$$

The conventional integrals are recovered from the complex integral by the transformations

$$t \to iu \qquad\qquad z \to ix \tag{48}$$

and are expressed by the equation

$$C(x) + iS(x) = \frac{1+i}{2} + \frac{1-i}{\sqrt{2}}\, E(ix) \tag{49}$$

The transformations

$$t \to -u^2 \qquad\qquad z \to -x^2 \tag{50}$$

give the error function of real argument

$$\pm H(x) = \frac{2}{\sqrt{\pi}} \int_0^{|x|} e^{-u^2}\, du = 1 - i\sqrt{2}E(-x^2) \tag{51}$$

where the $\pm$ sign is the same as the sign of x.

The transformations

$$t \to +u^2 \qquad\qquad z \to +x^2 \tag{52}$$

give the error function of imaginary argument

$$\mp iH(ix) = \frac{2}{\sqrt{\pi}} \int_0^{|x|} e^{u^2}\, du = i + \sqrt{2}E(+x^2) \tag{53}$$

where the $\mp$ sign is opposite to the sign of x.

The Stieltjes form of the Fresnel integral is given by the equation

$$E(z) = \frac{(\frac{1}{2}z)^{\frac{1}{2}} e^z}{\pi} \int_0^\infty \frac{e^{-u}}{u^{\frac{1}{2}}(z-u)}\, du \tag{54}$$

both sides of which approach zero as $z \to -\infty$, and have identically the same derivatives with respect to z.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{55}$$

or both of the inequalities

$$x^2 + y^2 < 1444 \qquad\qquad -x + 0.064\, y^2 \leqq 0 \tag{56}$$

then the Fresnel integral is computed from the ascending power series. Variation of the path of integration leads to the equation

$$E(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \frac{e^t}{t^{\frac{1}{2}}}\, dt + \frac{1}{\sqrt{2\pi}} \int_0^z \frac{e^t}{t^{\frac{1}{2}}}\, dt \tag{57}$$

Expansion in power series and term by term integration leads to the classical equation

$$E(z) = -\frac{i}{\sqrt{2}} + \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \sum_{m=0}^\infty \frac{z^m}{(2m+1)m!} \tag{58}$$

for which the constant of integration is given by the gamma function $\Gamma(\frac{1}{2})$ and is defined by the equation

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{2}}}\, dt = \sqrt{\pi} \tag{59}$$

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1444 \qquad\qquad -x + 0.064\, y^2 > 0 \tag{60}$$

then the Fresnel integral is computed from the rational approximation. When the converging factor is approximated by poles and residues, then the Fresnel integral is expressed by the equation

$$E(z) = \frac{z^{\frac{1}{2}} e^z}{\sqrt{2\pi}} \sum_{i=1}^{18} \frac{\epsilon_i}{z - \delta_i} \tag{61}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table II.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1444 \tag{62}$$

then the Fresnel integral is computed from the descending power series. Repeated integration by parts leads to the equation

$$E(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{m=0}^{N-1} \frac{(2m)!}{2^{2m} m! z^m} + \frac{(2N)!}{\sqrt{2\pi}\, 2^{2N} N!} \int_{-\infty}^{z} \frac{e^t}{t^{N+\frac{1}{2}}}\, dt \tag{63}$$

in which the series is asymptotic. Summation of the series is continued until there is no change in the sum or until $m = 37$.

Programming

SUBROUTINE CFRNLI (MO, AZ, FE)

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FORTRAN SUBROUTINE FOR COMPLEX FRESNEL INTEGRAL

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The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The complex Fresnel integral is computed by series expansions and rational approximations. If MO = 0, the real and imaginary parts of the function $E(z)$ are stored in array FE. If MO = 1, the real and imaginary parts of the function $e^{-z}E(z)$ are stored in array FE.

DOUBLE EXPONENTIAL INTEGRAL

Analysis

The double exponential integral $Di(z)$ is defined in terms of its argument z by the equation

$$Di(z) = \int_{-\infty}^{z} \frac{du}{u} \int_{-\infty}^{u} \frac{e^t}{t}\, dt \tag{64}$$

where the path of integration lies in that part of the complex plane from which the positive real axis is excluded.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{65}$$

or both of the inequalities

$$x^2 + y^2 < 2025 \qquad\qquad -x + 0.064\, y^2 \leqq 0 \tag{66}$$

then the double exponential integral is computed with an ascending power series. Substitution of $-t$ for t and integration by parts leads to the equation

$$\begin{aligned} Di(z) = \log(-z) \int_{-\infty}^{z} \frac{e^t}{t}\, dt \\ - \int_0^1 \frac{(1 - e^{-t})}{t} \log t\, dt + \int_1^{\infty} \frac{e^{-t}}{t} \log t\, dt \\ - \int_1^{-z} \frac{\log t}{t}\, dt + \int_0^{-z} \frac{(1 - e^{-t})}{t} \log t\, dt \end{aligned} \tag{67}$$

for which the constant of integration can be evaluated from the limiting value of the difference

$$\int_0^{\infty} t^{\nu-1} e^{-t} \log t\, dt - \int_0^1 t^{\nu-1} \log t\, dt \tag{68}$$

as $\nu \to 0$. Thus the reciprocal of ν is given by the equation

$$\frac{1}{\nu} = \int_0^1 t^{\nu-1}\, dt \tag{69}$$

and its derivative is given by the equation

$$-\frac{1}{\nu^2} = \int_0^1 t^{\nu-1} \log t\, dt \tag{70}$$

The gamma function $\Gamma(\nu)$ is given by the equation

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t}\, dt \tag{71}$$

and its derivative is given by the equation

$$\Gamma'(\nu) = \int_0^\infty t^{\nu-1} e^{-t} \log t\, dt \tag{72}$$

From page 236 of Reference 23, the gamma function is given also by the equation

$$\Gamma(\nu) = \frac{e^{-\gamma\nu}}{\nu} \prod_{n=1}^{\infty} \left[\frac{e^{\frac{\nu}{n}}}{1 + \frac{\nu}{n}} \right] \tag{73}$$

and its logarithmic derivative is given by the equation

$$\frac{\Gamma'(\nu)}{\Gamma(\nu)} = -\gamma - \frac{1}{\nu} + \nu \sum_{n=1}^{\infty} \frac{1}{n(n+\nu)} \tag{74}$$

The sum of the inverse squares of integers is given by the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{75}$$

Substitution of these formulae into the above difference between integrals shows that the limit as $\nu \to 0$ is expressed by the equation

$$\int_0^\infty \frac{e^{-t}}{t} \log t\, dt - \int_0^1 \frac{\log t}{t}\, dt = \frac{\pi^2}{12} + \frac{\gamma^2}{2} \tag{76}$$

Expansion in series and term by term integration leads to the equation

$$Di(z) = \frac{\pi^2}{12} + \tfrac{1}{2}(\gamma + \log(-z))^2 + \sum_{m=1}^{\infty} \frac{z^m}{m^2 m!} \tag{77}$$

in which the series is absolutely convergent.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 2025 \qquad\qquad -x + 0.064\, y^2 > 0 \tag{78}$$

then the double exponential integral is computed from the rational approximation. When the converging factor is approximated by poles and residues, then the double exponential integral is expressed by the equation

$$Di(z) = \frac{e^z}{z} \sum_{i=1}^{19} \frac{\epsilon_i}{z - \delta_i} \tag{79}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table III.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 2025 \tag{80}$$

then the double exponential integral is computed from the descending power series. Expansion of the integrand as a descending series and progressive integration by parts gives the approximation

$$Di(z) \sim \frac{e^z}{z^2} \sum_{m=0}^{N-1} \frac{(m+1)!}{z^m} \sum_{k=0}^{m} \frac{1}{k+1} \tag{81}$$

in which the series is asymptotic. Summation of the series is continued until there is no change in the sum or until $m = 44$.

Programming

SUBROUTINE CDEXPI (MO, AZ, FD)

```
****************************************************************************************************
FORTRAN SUBROUTINE FOR DOUBLE EXPONENTIAL INTEGRAL
****************************************************************************************************
```

The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The double exponential integral is computed by series expansions and rational approximations. If MO = 0, the real and imaginary parts of the function $Di(z)$ are stored in array FD. If MO = 1, the real and imaginary parts of the function $e^{-z}Di(z)$ are stored in array FD.

DOUBLE FRESNEL INTEGRAL

Analysis

The double Fresnel integral $D(z)$ is defined in terms of its argument z by the equation

$$D(z) = \int_{-\infty}^{z} \frac{du}{u} \int_{-\infty}^{u} \frac{e^t}{t^{\frac{1}{2}}}\, dt \tag{82}$$

where the path of integration lies within that part of the complex plane from which the positive real axis is excluded. The phase of z is limited to the range 0 to 2π, and the phase of $z^{1/2}$ is half the phase of z.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{83}$$

or both of the inequalities

$$x^2 + y^2 < 1849 \qquad\qquad -x + 0.064\, y^2 \leqq 0 \tag{84}$$

then the double Fresnel integral is computed from the ascending power series. Substitution of $-t$ for t and integration by parts leads to the equation

$$\begin{aligned} D(z) = \log(-z) \int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}}\, dt \\ + i \int_{0}^{\infty} \frac{e^{-t}}{t^{\frac{1}{2}}} \log t\, dt \\ - i \int_{0}^{-z} \frac{e^{-t}}{t^{\frac{1}{2}}} \log t\, dt \end{aligned} \tag{85}$$

for which the constant of integration is given by the equation

$$\Gamma'(\tfrac{1}{2}) = \int_{0}^{\infty} \frac{e^{-t}}{t^{\frac{1}{2}}} \log t\, dt = -(\gamma + 2\log 2)\sqrt{\pi} \tag{86}$$

Expansion in series and term by term integration leads to the equation

$$D(z) = -i(\gamma + 2\log 2 + \log(-z))\sqrt{\pi} + \sum_{m=0}^{\infty} \frac{z^{m+\frac{1}{2}}}{(m+\frac{1}{2})^2 m!} \tag{87}$$

in which the series is absolutely convergent.

If the argument $x+iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1849 \qquad -x + 0.064\,y^2 > 0 \tag{88}$$

then the double Fresnel integral is computed from the rational approximation. When the converging factor is approximated by poles and residues, then the double Fresnel integral is expressed by the equation

$$D(z) = \frac{e^z}{z^{\frac{1}{2}}} \sum_{i=1}^{19} \frac{\epsilon_i}{z - \delta_i} \tag{89}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table IV.

If the argument $x+iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1849 \tag{90}$$

then the double Fresnel integral is computed from the descending power series. Expansion of the integrand as a descending series and progressive integration by parts gives the approximation

$$D(z) \sim \frac{e^z}{z^{\frac{3}{2}}} \sum_{m=0}^{N-1} \frac{(2m+1)!}{2^{2m} m! z^m} \sum_{k=0}^{m} \frac{1}{2k+1} \tag{91}$$

in which the series is asymptotic. Summation of the series is continued until there is no change in the sum or until $m = 42$.

Programming

SUBROUTINE CDFRNI (MO, AZ, FD)

```
******************************************************************************************
FORTRAN SUBROUTINE FOR DOUBLE FRESNEL INTEGRAL
******************************************************************************************
```

The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The double Fresnel integral is computed by series expansions and rational approximations. If MO = 0, the real and imaginary parts of the function $D(z)$ are stored in array FD. If MO = 1, the real and imaginary parts of the function $e^{-z}D(z)$ are stored in array FD.

EXPONENTIAL EXPONENTIAL INTEGRAL

Analysis

The exponential exponential integral $k(z)$ is defined in terms of its argument z by the equation

$$k(z) = \int_{-\infty}^{z} \frac{e^{-u}}{u} \int_{-\infty}^{u} \frac{e^{t}}{t}\, dt\, du \tag{92}$$

where the path of integration lies in that part of the complex plane from which the positive real axis is excluded. Insofar as the exponential integral approaches its asymptotic approximation for large argument, the integrand and its integral approach zero along any circle of infinite radius which is centered at the origin. When the path of integration is deformed to follow partly an arc of infinite radius, the integration from $-\infty$ to z is converted into the negative of the integration from z to $+\infty$. The path of integration must not cross the positive real axis.

If the argument $x+iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{93}$$

then the exponential exponential integral is given by the equation

$$\begin{aligned} k(z) &= \tfrac{1}{4}\pi^2 + \tfrac{1}{2}(\gamma + \log(-z))^2 \\ &+ \sum_{m=1}^{\infty} \left(\gamma + \log(-z) - \frac{1}{m}\right)\frac{(-z)^m}{m \cdot m!} \\ &- \sum_{m=1}^{\infty} \frac{(-z)^m}{m \cdot m!} \sum_{k=1}^{m} \frac{1}{k} \end{aligned} \tag{94}$$

in which the series is convergent.

If the argument $x+iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1296 \qquad\qquad -x + 0.070\, y^2 \leqq 0 \tag{95}$$

then the exponential exponential integral is given by the equation

$$\begin{aligned} k(z) &= (\gamma + \log(-z))Ei(-z) \\ &- Di(-z) - e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k=m+1}^{\infty} \frac{1}{k^2} \end{aligned} \tag{96}$$

for which the integrals $Ei(-z)$ and $Di(-z)$ are computed from their rational approximations. Of two alternative series expansions the one given above has the advantage that the terms are all of the same sign on the positive real axis.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1296 \qquad\qquad - x + 0.070\, y^2 > 0 \tag{97}$$

then the exponential exponential integral is given by the equation

$$k(z) = - \sum_{i=1}^{18} \frac{\epsilon_i}{z - \delta_i} \tag{98}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table V.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1296 \tag{99}$$

then the exponential exponential integral is given by the approximation

$$k(z) \sim - \frac{1}{z} \sum_{m=0}^{N-1} \frac{m!}{(m+1) z^m} \tag{100}$$

in which the series is asymptotic. Summation of the series is continued until there is no change in the sum or until $m = 35$.

Programming

```
SUBROUTINE CEXEXI (AZ, FK)
****************************************************************************************************
FORTRAN SUBROUTINE FOR EXPONENTIAL EXPONENTIAL INTEGRAL
****************************************************************************************************
```

The real and imaginary parts of the argument z are given in array AZ. The exponential exponential integral is computed by series expansions and rational approximations. The real and imaginary parts of the function $k(z)$ are stored in array FK.

EXPONENTIAL FRESNEL INTEGRAL

Analysis

The exponential Fresnel integral $m(z)$ is defined in terms of its argument z by the equation

$$m(z) = \int_{-\infty}^{z} \frac{e^{-u}}{u^{\frac{1}{2}}} \int_{-\infty}^{u} \frac{e^{t}}{t}\, dt\, du \tag{101}$$

where the path of integration lies in that part of the complex plane from which the positive real axis is excluded. The phase of z is limited to the range 0 to 2π, and the phase of $z^{1/2}$ is half the phase of z. Insofar as the exponential integral approaches its asymptotic approximation for large argument, the integrand and its integral approach zero along any circle of infinite radius which is centered at the origin. When the path of integration is deformed to follow partly an arc of infinite radius, the integration from $-\infty$ to z is converted into the negative of the integration from z to $+\infty$. The path of integration must not cross the positive real axis.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{102}$$

then the exponential Fresnel integral is given by the equation

$$\begin{aligned} m(z) &= i\pi^{\frac{3}{2}} \\ &+ z^{\frac{1}{2}} \sum_{m=0}^{\infty} \left(\gamma + \log(-z) - \frac{1}{m+\frac{1}{2}}\right) \frac{(-z)^m}{(m+\frac{1}{2})m!} \\ &- z^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{(-z)^m}{(m+\frac{1}{2})m!} \sum_{k=1}^{m} \frac{1}{k} \end{aligned} \tag{103}$$

in which the series is convergent.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1296 \qquad\qquad -x + 0.070\, y^2 \leqq 0 \tag{104}$$

then the exponential Fresnel integral is given by the equation

$$\begin{aligned} m(z) &= \mp (\gamma + 2\log 2 + \log(-z))\sqrt{2\pi}\, iE(-z) \\ &\pm iD(-z) - e^{-z} \sum_{m=0}^{\infty} \frac{z^{m+\frac{1}{2}}}{\Gamma(m+\frac{3}{2})} \sum_{k=m+1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{k \cdot k!} \end{aligned} \tag{105}$$

for which the integrals $E(-z)$ and $D(-z)$ are computed from their rational approximations. The upper sign is valid when $y > 0$ and the lower sign is valid when $y \leqq 0$. Of two alternative series expansions the one given above has the advantage that the terms are all of the same sign on the positive real axis.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1296 \qquad\qquad - x + 0.070\, y^2 > 0 \tag{106}$$

then the exponential Fresnel integral is given by the equation

$$m(z) = -\, 2z^{\frac{1}{2}} \sum_{i=1}^{18} \frac{\epsilon_i}{z - \delta_i} \tag{107}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table VI.
If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1296 \tag{108}$$

then the exponential Fresnel integral is given by the approximation

$$m(z) \sim -\, \frac{1}{z^{\frac{1}{2}}} \sum_{m=0}^{N-1} \frac{m!}{(m + \frac{1}{2}) z^m} \tag{109}$$

in which the series is asymptotic. Summation of the series is continued until there is no change in the sum or until $m = 35$.

Programming

SUBROUTINE CEXFRI (AZ, FM)

```
****************************************************************************************************
FORTRAN SUBROUTINE FOR EXPONENTIAL FRESNEL INTEGRAL
****************************************************************************************************
```

The real and imaginary parts of the argument z are given in array AZ. The exponential Fresnel integral is computed by series expansions and rational approximations. The real and imaginary parts of the function $m(z)$ are stored in array FM.

LOGARITHMIC EXPONENTIAL INTEGRAL

Analysis

The logarithmic exponential integral $K(z, q)$ is defined by the equation

$$K(z, q) = \int_{-\infty}^{z} \frac{e^t}{t} \log\left(1 + \frac{t}{q}\right) dt \tag{110}$$

in which z is the argument and q is a parameter. The integrand is analytic at $t = 0$ but has a singularity at $t = -q$. The complex plane is cut along a line which extends from the singularity to infinite distance in a radial direction.

The logarithm for complex argument may be defined by the equation

$$\log\left(1 + \frac{t}{q}\right) = \int_0^t \frac{du}{q + u} \tag{111}$$

where the path of integration extends radially outward from the origin.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{112}$$

or both of the inequalities

$$x^2 + y^2 < 1600 \qquad\qquad - x + 0.064\, y^2 \leqq 0 \tag{113}$$

then the logarithmic exponential integral is computed from an ascending power series. The path of integration may be varied to pass through the origin, and the exponential function in the integrand may be expanded in a power series.

The constant of integration is given by the equation

$$\int_{-\infty}^{0} \frac{e^t}{t} \log\left(1 + \frac{t}{q}\right) dt = - \int_{-\infty}^{q} \frac{e^{-u}}{u} \int_{-\infty}^{u} \frac{e^t}{t}\, dt\, du \tag{114}$$

both sides of which approach zero as $q \to -\infty$, and have identically the same derivatives with respect to q.

The logarithmic exponential integral is given by the equation

$$K(z, q) = - k(q) + L\left(1 + \frac{z}{q}\right) + \sum_{m=1}^{\infty} T_m \tag{115}$$

for which the terms are defined by the equation

$$T_m = \frac{1}{m \cdot m!}\left\{z^m \log\left(1 + \frac{z}{q}\right) - \int_0^z \frac{u^m}{q + u}\, du\right\} \tag{116}$$

When $|q| \leqq \sqrt{1.3}|z|$ the series is started with the equation

$$T_1 = (q + z) \log\left(1 + \frac{z}{q}\right) - z \tag{117}$$

and is continued in the direction of ascending order with the recurrence equation

$$T_m = \frac{z^{m-1}}{m \cdot m!}\left\{(q + z) \log\left(1 + \frac{z}{q}\right) - \frac{z}{m}\right\} - \frac{(m-1)}{m^2} q T_{m-1} \tag{118}$$

The recurrence is cycled to the order N to which the ascending series of the exponential integral must be carried for accurate evaluation.

When $|q| > \sqrt{1.3}|z|$ the terms are given by the equation

$$T_m = -\frac{z^m}{m!} \sum_{k=1}^{\infty} \frac{1}{k(m+k)}\left(-\frac{z}{q}\right)^k \tag{119}$$

The series is started at the order N and is continued in the direction of descending order with the recurrence equation

$$T_m = \frac{z^m}{m \cdot m!}\left\{\left(1 + \frac{z}{q}\right) \log\left(1 + \frac{z}{q}\right) - \frac{1}{m+1} \frac{z}{q}\right\} - \frac{(m+1)^2}{mq} T_{m+1} \tag{120}$$

The terms in the expansion of the exponential integral are computed in advance of recurrence and are stored in an array.

An integration by parts leads to the equation

$$\int_{-\infty}^{z} \frac{e^t}{t} \log\left(1 + \frac{t}{q}\right) dt = \log\left(1 + \frac{z}{q}\right) \int_{-\infty}^{z} \frac{e^t}{t}\, dt - \int_{-\infty}^{z} \frac{du}{q+u} \int_{-\infty}^{u} \frac{e^t}{t}\, dt \tag{121}$$

If the modulus of z is large enough, the exponential integral can be replaced by its rational or asymptotic approximations. The path of integration must lie outside of a circle of radius $|z|$. If this path would cross a cut, then the actual integrand must be integrated in along the cut, around the singularity, and out along the cut. To the integration along the outer path must be added the correction

$$\pm 2\pi i \int_{-\infty}^{z} \frac{e^t}{t}\, dt \mp 2\pi i \int_{-\infty}^{-q} \frac{e^t}{t}\, dt \tag{122}$$

where the upper sign is used when the singularity is above the real axis and the lower sign is used when the singularity is on or below the real axis.

In the integration along the outer path the imaginary part of the logarithm does not reverse sign where the path crosses the cut. If the logarithm is given its usual value at the end of integration, then an additional correction in the amount of

$$\mp 2\pi i \int_{-\infty}^{z} \frac{e^t}{t}\, dt \tag{123}$$

must be added to the integration along the outer path. Only the integration to $-q$ remains when both corrections are added.

A crossing of the cut occurs only if $|z| > |q|$ and the inequalities

$$\mathcal{I}m\, q < 0 \qquad \mathcal{I}m\, z > 0 \qquad \mathcal{I}m \log\left(1 + \frac{z}{q}\right) > 0 \tag{124}$$

or the inequalities

$$\mathcal{I}m\, q > 0 \qquad \mathcal{I}m\, z \leqq 0 \qquad \mathcal{I}m \log\left(1 + \frac{z}{q}\right) < 0 \tag{125}$$

also are satisfied. If q happens to lie on the positive real axis, then a crossing occurs if the inequality

$$\mathcal{I}m \log\left(1 + \frac{z}{q}\right) \leqq 0 \tag{126}$$

is satisfied.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1600 \qquad - x + 0.064\, y^2 > 0 \tag{127}$$

then the logarithmic exponential integral is computed from the rational approximation for the exponential integral. Along the outer path the contribution to the integration is

$$K(z,q) \sim \sum_{i=1}^{18} \epsilon_i \left[\frac{e^z}{z - \delta_i} \log\left(1 + \frac{z}{q}\right) - \frac{e^{\delta_i}}{q + \delta_i} \int_{-\infty}^{z-\delta_i} \frac{e^t}{t}\, dt + \frac{e^{-q}}{q + \delta_i} \int_{-\infty}^{q+z} \frac{e^t}{t}\, dt \right] \tag{128}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table I.

If the outer path crosses the cut, and also ends closer to the real axis than the singularity, or if the outer path does not cross the cut, yet ends farther from the real axis than the singularity, then a correction must be added to that exponential integral whose argument is $q + z$. The correction is

$$\pm\, 2\pi i \tag{129}$$

where the sign is determined by the direction in which the singularity must be encircled in order to correct the evaluation of the exponential integral. If $|z| > |q|$, then the lower sign is used when the singularity is above the real axis and $\mathcal{I}m(q + z) \leqq 0$, but the upper sign is used when the singularity is below the real axis and $\mathcal{I}m(q + z) > 0$. If $|z| \leqq |q|$, then the signs are reversed.

If the arguments satisfy both of the inequalities

$$|q + \delta_i| \leqq \tfrac{1}{2} \qquad |q + \delta_i| \leqq \tfrac{1}{2}|q + z| \tag{130}$$

then the last two integrals in the rational approximation are nearly equal. Their difference can be expressed by a series expansion in accordance with the equation

$$\frac{1}{q + \delta_i} \left[e^{-(z-\delta_i)} \int_{-\infty}^{z-\delta_i} \frac{e^t}{t}\, dt - e^{-(q+z)} \int_{-\infty}^{q+z} \frac{e^t}{t}\, dt \right] = \sum_{n=0}^{\infty} T_n \tag{131}$$

where the term of lowest order is given by the equation

$$T_0 = -\frac{1}{q+z} + e^{-(q+z)} \int_{-\infty}^{q+z} \frac{e^t}{t}\, dt \tag{132}$$

and the terms of higher order are given by the equations

$$T_n = -\frac{1}{(n+1)(q+z)}\left(\frac{q+\delta_i}{q+z}\right)^n + \frac{(q+\delta_i)}{(n+1)}\, T_{n-1} \tag{133}$$

The recurrence is cycled until there is no change in the summation.

If the argument $x+iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1600 \tag{134}$$

then the logarithmic exponential integral is computed from the asymptotic approximation for the exponential integral. Along the outer path the contribution to the integration is

$$K(z,q) \sim e^z \sum_{n=1}^{N-1} T_n \tag{135}$$

where the terms are defined by the equation

$$T_n = \frac{(n-1)!}{z^n} \log\left(1 + \frac{z}{q}\right) - (n-1)!e^{-z} \int_{-\infty}^{z} \frac{e^u}{u^n(q+u)}\, du \tag{136}$$

When $|z| \leqq \sqrt{6}|q|$ the series is started with the equation

$$T_1 = \frac{1}{z} \log\left(1 + \frac{z}{q}\right) - \frac{1}{q}\, e^{-z} \int_{-\infty}^{z} \frac{e^t}{t}\, dt + \frac{1}{q}\, e^{-(q+z)} \int_{-\infty}^{q+z} \frac{e^t}{t}\, dt \tag{137}$$

and is continued in the direction of descending order with the recurrence equation

$$T_n = \frac{(n-1)!}{z^n}\left(1 + \frac{z}{q}\right) \log\left(1 + \frac{z}{q}\right) - \frac{(n-1)!}{q}\, e^{-z} \int_{-\infty}^{z} \frac{e^t}{t^n}\, dt - \frac{(n-1)}{q}\, T_{n-1} \tag{138}$$

When $|z| > \sqrt{6}|q|$ the series is started at the order N and is continued in the direction of ascending order with the recurrence equation

$$T_n = \frac{(n-1)!}{z^{n+1}}\,(q+z) \log\left(1 + \frac{z}{q}\right) - (n-1)!\, e^{-z} \int_{-\infty}^{z} \frac{e^t}{t^{n+1}}\, dt - \frac{q}{n}\, T_{n+1} \tag{139}$$

In either case the exponential integral is generated with the recurrence equation

$$(n-1)!e^{-z} \int_{-\infty}^{z} \frac{e^t}{t^n}\, dt = \frac{(n-1)!}{z^n} + n!e^{-z} \int_{-\infty}^{z} \frac{e^t}{t^{n+1}}\, dt \tag{140}$$

The terms for the asymptotic expansion are computed in advance of recurrence and are stored in an array. The maximum order N is determined by whichever of the criteria

$$\frac{N!}{|z|^N} << 1 \qquad\qquad \left|\frac{q}{z}\right|^N << 1 \tag{141}$$

is appropriate for accurate computation.

When $q + z \equiv 0$, the logarithmic exponential integral is taken to be that value which is approached when z/q is real and approaches -1.

Programming

SUBROUTINE CLGEXI (MO, AZ, CQ, FK)

```
****************************************************************************************
FORTRAN SUBROUTINE FOR LOGARITHMIC EXPONENTIAL INTEGRAL
****************************************************************************************
```

The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The real and imaginary parts of the parameter q are given in array CQ. The logarithmic exponential integral is computed by series expansions and rational approximations. Calls are made to Subroutine CLGMCI, Subroutine CEXPLI, and Subroutine CEXEXI. If MO = 0, the real and imaginary parts of the function $K(z, q)$ are stored in array FK. If MO = 1, the real and imaginary parts of the function $e^{-z}K(z, q)$ are stored in array FK.

ARCTANGENTIAL FRESNEL INTEGRAL

Analysis

The arctangential Fresnel integral $M(z, q)$ is defined by the equation

$$M(z, q) = \int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} dt \tag{142}$$

in which z is the argument and q is a parameter. The integrand is analytic at $t = 0$ but has a singularity at $t = -q$. The complex plane is cut along a line which extends from the singularity to infinite distance in a radial direction.

The arctangent for complex argument may be defined by the equation

$$\tan^{-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} = \tfrac{1}{2} q^{\frac{1}{2}} \int_{0}^{t} \frac{du}{u^{\frac{1}{2}}(q + u)} \tag{143}$$

where the path of integration extends radially outward from the origin. Resolution of the integrand into partial fractions and integration leads to the equation

$$\tan^{-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} = \tfrac{1}{2} i \log \left\{1 - i\left(\frac{t}{q}\right)^{\frac{1}{2}}\right\} - \tfrac{1}{2} i \log \left\{1 + i\left(\frac{t}{q}\right)^{\frac{1}{2}}\right\} \tag{144}$$

which is useful for numerical evaluation.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{145}$$

or both of the inequalities

$$x^2 + y^2 < 1444 \qquad\qquad - x + 0.064\, y^2 \leqq 0 \tag{146}$$

then the arctangential Fresnel integral is computed from an ascending power series. The path of integration may be varied to pass through the origin, and the exponential function in the integrand may be expanded in a power series.

The constant of integration is given by the equation

$$\int_{-\infty}^{0} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} dt = -\tfrac{1}{2} \int_{-\infty}^{q} \frac{e^{-u}}{u^{\frac{1}{2}}} \int_{-\infty}^{u} \frac{e^t}{t}\, dt\, du \tag{147}$$

both sides of which approach zero as $q \to -\infty$ and have identically the same derivatives with respect to q.

The arctangential Fresnel integral is given by the equation

$$M(z, q) = -\tfrac{1}{2}m(q) + \sum_{m=0}^{\infty} T_m \tag{148}$$

for which the terms are defined by the equation

$$T_m = \frac{1}{(m+\frac{1}{2})m!}\left\{z^{m+\frac{1}{2}} \tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} - \tfrac{1}{2}q^{\frac{1}{2}} \int_0^z \frac{u^m}{q+u}\, du\right\} \tag{149}$$

When $|q| \leqq \sqrt{1.3}|z|$ the series is started with the equation

$$T_0 = -\,(q^{\frac{1}{2}} - iz^{\frac{1}{2}}) \log\left\{1 - i\left(\frac{z}{q}\right)^{\frac{1}{2}}\right\} - (q^{\frac{1}{2}} + iz^{\frac{1}{2}}) \log\left\{1 + i\left(\frac{z}{q}\right)^{\frac{1}{2}}\right\} \tag{150}$$

and is continued in the direction of ascending order with the recurrence equation

$$T_m = \frac{z^{m-\frac{1}{2}}}{(m+\frac{1}{2})m!}\left\{(q+z)\tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{q}{2m}\left(\frac{z}{q}\right)^{\frac{1}{2}}\right\} - \frac{(m-\frac{1}{2})}{(m+\frac{1}{2})}\frac{q}{m}\, T_{m-1} \tag{151}$$

The recurrence is cycled to the order N to which the ascending series of the Fresnel integral must be carried for accurate evaluation.

When $|q| > \sqrt{1.3}|z|$ the terms are given by the equation

$$T_m = -\tfrac{1}{2}q^{\frac{1}{2}}\frac{z^m}{m!}\sum_{k=1}^{\infty}\frac{1}{(k-\frac{1}{2})(m+k)}\left(-\frac{z}{q}\right)^k \tag{152}$$

The series is started at the order N and is continued in the direction of descending order with the recurrence equation

$$T_m = \frac{z^{m+\frac{1}{2}}}{(m+\frac{1}{2})m!}\left\{\left(1+\frac{z}{q}\right)\tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{\frac{1}{2}}{m+1}\left(\frac{z}{q}\right)^{\frac{1}{2}}\right\} - \frac{(m+\frac{3}{2})}{(m+\frac{1}{2})}\frac{(m+1)}{q}\, T_{m+1} \tag{153}$$

The terms in the expansion of the Fresnel integral are computed in advance of recurrence and are stored in an array.

An integration by parts leads to the equation

$$\int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} dt = \tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} \int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}}\, dt - \tfrac{1}{2}q^{\frac{1}{2}} \int_{-\infty}^{z} \frac{du}{u^{\frac{1}{2}}(q+u)} \int_{-\infty}^{u} \frac{e^t}{t^{\frac{1}{2}}}\, dt \tag{154}$$

If the modulus of z is large enough, the Fresnel integral can be replaced by its rational or asymptotic approximations. The path of integration must lie outside of a circle of radius $|z|$. If this path would cross a cut, then the actual integrand must be integrated in along the cut, around the singularity, and out along the cut. To the integration along the outer path must be added the correction

$$\mp\, \pi \int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}}\, dt \pm \pi \int_{-\infty}^{-q} \frac{e^t}{t^{\frac{1}{2}}}\, dt \tag{155}$$

where the lower sign is used when the singularity is on the negative real axis, but the upper sign is used when the singularity is anywhere else.

In the integration along the outer path the real part of the arctangent does not reverse sign where the path crosses the cut. If the arctangent is given its usual value at the end of integration, then an additional correction in the amount of

$$\pm \pi \int_{-\infty}^{z} \frac{e^t}{t^{\frac{1}{2}}}\, dt \tag{156}$$

must be added to the integration along the outer path. Only the integration to $-q$ remains when both corrections are added.

A crossing of the cut occurs only if $|z| > |q|$ and the inequalities

$$\mathcal{I}m\, q < 0 \qquad \mathcal{I}m\, z > 0 \qquad \mathcal{R}e \tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} < 0 \tag{157}$$

or the inequalities

$$\mathcal{I}m\, q > 0 \qquad \mathcal{I}m\, z \leqq 0 \qquad \mathcal{R}e \tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} < 0 \tag{158}$$

also are satisfied. If q happens to lie on the positive real axis, then a crossing occurs if the inequality

$$\mathcal{R}e \tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} \geqq 0 \tag{159}$$

is satisfied.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 1444 \qquad - x + 0.064\, y^2 > 0 \tag{160}$$

then the arctangential Fresnel integral is computed from the rational approximation for the Fresnel integral. Along the outer path the contribution to the integration is

$$M(z, q) \sim \sum_{i=1}^{18} \epsilon_i \left[\frac{z^{\frac{1}{2}} e^z}{z - \delta_i} \tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{\frac{1}{2} q^{\frac{1}{2}} e^{\delta_i}}{q + \delta_i} \int_{-\infty}^{z-\delta_i} \frac{e^t}{t}\, dt + \frac{\frac{1}{2} q^{\frac{1}{2}} e^{-q}}{q + \delta_i} \int_{-\infty}^{q+z} \frac{e^t}{t}\, dt \right] \tag{161}$$

for which the positions δ_i and the residues ϵ_i of the poles are listed in Table II.

If the outer path crosses the cut, and also ends closer to the real axis than the singularity, or if the outer path does not cross the cut, yet ends farther from the real axis than the singularity, then a correction must be added to that exponential integral whose argument is $q + z$. The correction is

$$\pm\, 2\pi i \tag{162}$$

where the sign is determined by the direction in which the singularity must be encircled in order to correct the evaluation of the exponential integral. If $|z| > |q|$, then the lower sign is used when the singularity is above the real axis and $\mathcal{I}m(q + z) \leqq 0$, but the upper sign is used when the singularity is below the real axis and $\mathcal{I}m(q + z) > 0$. If $|z| \leqq |q|$, then the signs are reversed.

If the arguments satisfy both of the inequalities

$$|q + \delta_i| \leqq \tfrac{1}{2} \qquad\qquad |q + \delta_i| \leqq \tfrac{1}{2}|q + z| \tag{163}$$

then the last two integrals in the rational approximation are nearly equal. Their difference can be expressed by a series expansion in accordance with the equation

$$\frac{1}{q+\delta_i}\left[e^{-(z-\delta_i)} \int_{-\infty}^{z-\delta_i} \frac{e^t}{t}\,dt - e^{-(q+z)} \int_{-\infty}^{q+z} \frac{e^t}{t}\,dt \right] = \sum_{n=0}^{\infty} T_n \tag{164}$$

where the term of lowest order is given by the equation

$$T_0 = -\frac{1}{q+z} + e^{-(q+z)} \int_{-\infty}^{q+z} \frac{e^t}{t}\,dt \tag{165}$$

and the terms of higher order are given by the equations

$$T_n = -\frac{1}{(n+1)(q+z)}\left(\frac{q+\delta_i}{q+z}\right)^n + \frac{(q+\delta_i)}{(n+1)}\,T_{n-1} \tag{166}$$

The recurrence is cycled until there is no change in the summation.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 1444 \tag{167}$$

then the arctangential Fresnel integral is computed from the asymptotic approximation for the Fresnel integral. Along the outer path the contribution to the integration is

$$M(z, q) \sim e^z \sum_{n=1}^{N-1} T_n \tag{168}$$

where the terms are defined by the equation

$$T_n = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})z^{n-\frac{1}{2}}} \tan^{-1}\!\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\,\tfrac{1}{2}q^{\frac{1}{2}}e^{-z} \int_{-\infty}^{z} \frac{e^u}{u^n(q+u)}\,du \tag{169}$$

When $|z| \leqq \sqrt{6}|q|$ the series is started with the equation

$$T_1 = \frac{1}{z^{\frac{1}{2}}} \tan^{-1}\!\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{\frac{1}{2}}{q^{\frac{1}{2}}}\, e^{-z} \int_{-\infty}^{z} \frac{e^t}{t}\,dt + \frac{\frac{1}{2}}{q^{\frac{1}{2}}}\, e^{-(q+z)} \int_{-\infty}^{q+z} \frac{e^t}{t}\,dt \tag{170}$$

and is continued in the direction of descending order with the recurrence equations

$$T_n = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})z^{n-\frac{1}{2}}}\left(1 + \frac{z}{q}\right) \tan^{-1}\!\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\,\frac{\frac{1}{2}}{q^{\frac{1}{2}}}\, e^{-z} \int_{-\infty}^{z} \frac{e^t}{t^n}\,dt - \frac{(n-\frac{3}{2})}{q}\,T_{n-1} \tag{171}$$

When $|z| > \sqrt{6}|q|$ the series is started at the order N and is continued in the direction of ascending order with the recurrence equation

$$T_n = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})z^{n+\frac{1}{2}}}\,(q+z)\tan^{-1}\left(\frac{z}{q}\right)^{\frac{1}{2}} - \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\,\tfrac{1}{2}q^{\frac{1}{2}}e^{-z}\int_{-\infty}^{z}\frac{e^t}{t^{n+1}}\,dt - \frac{q}{n-\frac{1}{2}}\,T_{n+1} \qquad (172)$$

In either case the exponential integral is generated with the recurrence equation

$$\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}\,e^{-z}\int_{-\infty}^{z}\frac{e^t}{t^n}\,dt = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})z^n} + \frac{n}{(n-\frac{1}{2})}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,e^{-z}\int_{-\infty}^{z}\frac{e^t}{t^{n+1}}\,dt \qquad (173)$$

The terms for the asymptotic expansion are computed in advance of recurrence and are stored in an array. The maximum order N is determined by whichever of the criteria

$$\frac{\Gamma(N+\frac{1}{2})}{\Gamma(\frac{1}{2})|z|^N} << 1 \qquad\qquad \left|\frac{q}{z}\right|^N << 1 \qquad (174)$$

is appropriate for accurate computation.

When $q + z \equiv 0$, the arctangential Fresnel integral is taken to be that value which is approached when z/q is real and approaches -1.

Programming

SUBROUTINE CATFRI (MO, AZ, CQ, FM)

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FORTRAN SUBROUTINE FOR ARCTANGENTIAL FRESNEL INTEGRAL

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The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ. The real and imaginary parts of the parameter q are given in array CQ. The arctangential Fresnel integral is computed by series expansions and rational approximations. Calls are made to Subroutine CEXPLI, Subroutine CFRNLI, and Subroutine CEXFRI. If MO = 0, the real and imaginary parts of the function $M(z, q)$ are stored in array FM. If MO = 1, the real and imaginary parts of the function $e^{-z}M(z, q)$ are stored in array FM.

Analysis

The Bessel function $J_\nu(z)$ is given by the equation

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)} \tag{175}$$

while the Weber function $Y_\nu(z)$ is given by the equation

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \quad . \tag{176}$$

The largest term in the ascending series for $J_\nu(z)$ occurs where

$$\frac{|\frac{1}{2}z|^2}{m|\nu+m|} \sim 1 \tag{177}$$

The magnitude of the largest term is given by the Stirling approximation

$$\frac{|\frac{1}{2}z|^{\nu+2m}}{m!\Gamma(\nu+m+1)} \sim \frac{1}{2\pi\sqrt{m(\nu+m)}} \left(\frac{m}{\nu+m}\right)^{\frac{\nu}{2}} e^{\nu+2m} \tag{178}$$

which remains bounded as $\nu \to \infty$ if the ratio between m and ν does not exceed a maximum limit. The maximum ratio α satisfies the transcendental equation

$$\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}} e^{1+2\alpha} = 1 \tag{179}$$

whose solution is

$$\alpha = 9.983932012886692\times10^{-2} \tag{180}$$

The order and the argument are bounded by the limiting relations

$$m = \alpha\nu \qquad\qquad |z| = \mu\nu \tag{181}$$

for which the ratio μ is given by the equation

$$\mu = 2(\alpha+\alpha^2)^{\frac{1}{2}} = 0.6627434193491816 \tag{182}$$

If ν is replaced by $-\nu$, then α is replaced by $-1-\alpha$, while μ remains the same. Thus the maximum term remains bounded if the order and the argument satisfy the inequality

$$|z| \leq \mu|\nu| \tag{183}$$

A popular method for the computation of Bessel functions in the critical region which cannot be reached by the ascending series utilizes the recurrence equations

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) \tag{184}$$

$$Y_{\nu-1}(z) + Y_{\nu+1}(z) = \frac{2\nu}{z} Y_\nu(z) \tag{185}$$

which are cycled in the direction of descending order. During the μth cycle a rounding error ϵ_μ is introduced and this rounding error then persists in subsequent cycles. In view of the recurrence relation

$$J_\nu(z)Y_{\nu+1}(z) - J_{\nu+1}(z)Y_\nu(z) = -\frac{2}{\pi z} \tag{186}$$

the persisting error in the νth cycle is given by the expression

$$-\frac{\pi z}{2}\left\{Y_{\mu+1}(z)J_\nu(z) - J_{\mu+1}(z)Y_\nu(z)\right\}\epsilon_\mu \tag{187}$$

for a descending recurrence. Interchange of μ and ν in the coefficient of the expression gives the persisting error for an ascending recurrence. The recurrence can be cycled in either direction without decay or growth of error wherever $|J_\nu(z)|$ and $|Y_\nu(z)|$ both are bounded. The practical range for recurrence has been determined by actual computation to be where $|z| > 0.8|\nu|$. Elsewhere the recurrence tends to accentuate whichever of the functions is increasing relative to the other. A large number of cycles is required when the order is small but the argument is large.

If the order and the argument satisfy the inequality

$$|z| \geqq 17.5 + \tfrac{1}{2}|\nu|^2 \tag{188}$$

then the Bessel function is given by the equation

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left\{P_\nu(z)\cos\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) - Q_\nu(z)\sin\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right)\right\} \tag{189}$$

where the function $P_\nu(z)$ is the sum of the even-ordered terms and the function $Q_\nu(z)$ is the sum of the odd-ordered terms in the equation

$$P_\nu(z) + iQ_\nu(z) = \sum_{m=0}^{N-1} \frac{\Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu - m + \frac{1}{2})(-2iz)^m} \tag{190}$$

The smallest term in the descending series occurs where

$$\frac{(m - \frac{1}{2})^2 - \nu^2}{m|2z|} \sim 1 \tag{191}$$

The magnitude of the smallest term is given by the Stirling approximation

$$\left(\frac{2}{\pi m}\right)^{\frac{1}{2}}\left(\frac{m + \nu - \frac{1}{2}}{m - \nu - \frac{1}{2}}\right)^{\nu} e^{1-m} \tag{192}$$

which remains bounded as $\nu \to \infty$ if the ratio between m and ν exceeds a minimum

limit. The minimum ratio β satisfies the transcendental equation

$$\left(\frac{\beta+1}{\beta-1}\right)e^{-\beta} = 1 \tag{193}$$

whose solution is

$$\beta = 1.543404638418208 \tag{194}$$

The order and the argument are bounded by the limiting relations

$$m = \beta\nu \qquad\qquad |z| = \mu\nu \tag{195}$$

for which the ratio μ is given by the equation

$$\mu = \frac{(\beta^2-1)}{2\beta} = 0.4477432046943028 \tag{196}$$

This value is far below the range of ratios for actual use of the asymptotic series, which can be cycled until there is no change in the sum.

Programming

SUBROUTINE CBSSLJ (AZ, CN, FJ)

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*******************************************************************************************
FORTRAN SUBROUTINE FOR ORDINARY BESSEL FUNCTION OF FIRST KIND
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The real and imaginary parts of the argument z are given in array AZ, and the real and imaginary parts of the order ν are given in array CN. The complex Bessel function is computed by series expansions and recurrence relations. Calls are made to Subroutine CGAMMA. The real and imaginary parts of the function $J_\nu(z)$ are stored in array FJ.

MODIFIED BESSEL FUNCTION

Analysis

In the complex plane a majority of the Bessel functions are multiple valued. A variety of Bessel functions for real argument can be expressed in terms of the modified Bessel function $K_\nu(z)$ for complex argument. According to Equation (8), page 78 of Reference 24, the Hankel function of the first kind is given by the equation

$$H_\nu^{(1)}(z) = -\frac{2}{\pi} i e^{-\frac{1}{2}\nu\pi i} K_\nu(-iz) \tag{197}$$

According to Equation (5), page 75 of Reference 24, the Hankel functions are related by the equation

$$H_\nu^{(1)}(ze^{\pi i}) = -e^{-\nu\pi i} H_\nu^{(2)}(z) \tag{198}$$

and the Hankel function of the second kind is given by the equation

$$H_\nu^{(2)}(z) = \frac{2}{\pi} i e^{\frac{1}{2}\nu\pi i} K_\nu(iz) \tag{199}$$

Since the Bessel functions are related to the Hankel functions by the equation

$$J_\nu(z) = \frac{1}{2}\left\{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right\} \tag{200}$$

the Bessel functions are given by the equation

$$J_\nu(z) = \frac{i}{\pi}\left\{e^{\frac{1}{2}\nu\pi i} K_\nu(iz) - e^{-\frac{1}{2}\nu\pi i} K_\nu(-iz)\right\} \tag{201}$$

and since the Weber functions are related to the Hankel functions by the equation

$$Y_\nu(z) = \frac{1}{2i}\left\{H_\nu^{(1)}(z) - H_\nu^{(2)}(z)\right\} \tag{202}$$

the Weber functions are given by the equation

$$Y_\nu(z) = -\frac{1}{\pi}\left\{e^{\frac{1}{2}\nu\pi i} K_\nu(iz) + e^{-\frac{1}{2}\nu\pi i} K_\nu(-iz)\right\} \tag{203}$$

According to Equation (18), page 80 of Reference 24, the modified Bessel function $I_\nu(z)$ is given by the equation

$$I_\nu(z) = \frac{i}{\pi}\left\{K_\nu(ze^{\pi i}) - e^{-\nu\pi i} K_\nu(z)\right\} \tag{204}$$

or by the equation

$$I_\nu(z) = -\frac{i}{\pi}\left\{K_\nu(ze^{-\pi i}) - e^{\nu\pi i} K_\nu(z)\right\} \tag{205}$$

Elimination of $I_\nu(z)$ from these two equations gives the equation

$$K_\nu(ze^{\pi i}) + K_\nu(ze^{-\pi i}) = 2\cos\nu\pi\, K_\nu(z) \tag{206}$$

which is useful for extending the range of argument beyond the usual limits.

The Airy function is defined by the equation

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \tfrac{1}{3}t^3\right) dt \tag{207}$$

The function $Ai(x)$ is expressed by the equation

$$Ai(x) = \frac{1}{\pi}\left(\tfrac{1}{3}x\right)^{\frac{1}{2}} K_{\frac{1}{3}}\left(\tfrac{2}{3}x^{\frac{3}{2}}\right) \tag{208}$$

and its derivative $Ai'(x)$ is given by the equation

$$Ai'(x) = -\frac{x}{\pi\sqrt{3}}\, K_{\frac{2}{3}}\left(\tfrac{2}{3}x^{\frac{3}{2}}\right) \tag{209}$$

The Airy function is useful for the approximation of Bessel functions when the argument and the order are large and nearly equal. They have applications in problems of stationary phase.

On the positive real axis $K_\nu(z)$ is real and has a critical region where it cannot be evaluated easily by series expansions. It is a natural choice for a rational polynomial approximation in the complex plane. The argument z is restricted to that part of the complex plane from which the negative real axis is excluded. The phase of z is limited to the range $-\pi$ to $+\pi$.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \leqq 1 \tag{210}$$

or both of the inequalities

$$x^2 + y^2 < 289 \qquad\qquad + x + 0.096\, y^2 \leqq 0 \tag{211}$$

then the Bessel functions are computed from the ascending power series. The function $I_\nu(z)$ is given for any order ν by the equation

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \tag{212}$$

whence the function $K_\nu(z)$ is given for fractional order by the equation

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi} \tag{213}$$

The function $K_\nu(z)$ is given for integral order by the equation

$$K_n(z) = (-1)^{n+1} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2m}}{m!(n+m)!} \left\{ \gamma + \log\left(\tfrac{1}{2}z\right) - \frac{1}{2}\sum_{k=1}^{m}\frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n+m}\frac{1}{k} \right\}$$
$$+ \frac{1}{2}\sum_{m=0}^{n-1} (-1)^m \frac{(n-m-1)!}{m!} \left(\tfrac{1}{2}z\right)^{-n+2m} \tag{214}$$

where γ is Euler's constant. The summations are continued until there is no change in the associated value of $I_\nu(z)$.

If the argument $x + iy$ satisfies both of the inequalities

$$1 < x^2 + y^2 < 289 \qquad\qquad + x + 0.096\, y^2 > 0 \tag{215}$$

then the Bessel function is computed from the rational approximation. When the converging factor is approximated by poles and residues, then the Bessel function is expressed by the equation

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left\{ 1 + \sum_{i=1}^{14} \frac{\epsilon_i}{z - \delta_i} \right\} \tag{216}$$

for which the positions δ_i and the residues ϵ_i, for orders 0, $\frac{1}{3}$, $\frac{2}{3}$, 1, are listed in Tables VII through X.

If the argument $x + iy$ satisfies the inequality

$$x^2 + y^2 \geqq 289 \tag{217}$$

then the Bessel functions are given by descending power series. According to Equation (4), page 172 of Reference 24, the integral representation of the function $K_\nu(z)$ is given by the equation

$$K_\nu(z) = \frac{\Gamma(\frac{1}{2})(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\nu - \frac{1}{2}}\, dt \tag{218}$$

which may be transformed through the substitution

$$t = 1 + \frac{u}{z} \tag{219}$$

to give the equation

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \frac{e^{-z}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-u} u^{\nu - \frac{1}{2}} \left(1 + \frac{u}{2z}\right)^{\nu - \frac{1}{2}} du \tag{220}$$

Expansion of the integrand with the binomial series

$$\left(1 + \frac{u}{2z}\right)^{\nu - \frac{1}{2}} = \sum_{m=0}^{N-1} \frac{\Gamma(\nu + \frac{1}{2})}{m!\,\Gamma(\nu - m + \frac{1}{2})} \left(\frac{u}{2z}\right)^m$$
$$+ \frac{\Gamma(\nu + \frac{1}{2})}{(N-1)!\,\Gamma(\nu - N + \frac{1}{2})} \left(\frac{u}{2z}\right)^N \int_0^1 \left(1 + \frac{ut}{2z}\right)^{\nu - N - \frac{1}{2}} (1 - t)^{N-1}\, dt \tag{221}$$

and term by term integration leads to the equation

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{m=0}^{N-1} \frac{\Gamma(\nu+m+\frac{1}{2})}{m!\Gamma(\nu-m+\frac{1}{2})(2z)^m}$$

$$+ \frac{\pi^{\frac{1}{2}} e^{-z}}{(N-1)!(2z)^{N+\frac{1}{2}}} \int_0^\infty \int_0^1 e^{-u} u^{\nu+N-\frac{1}{2}} \left(1+\frac{ut}{2z}\right)^{\nu-N-\frac{1}{2}} (1-t)^{N-1} dt\, du \tag{222}$$

in which the series is asymptotic. The function is given by the equivalent equation

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left\{ \sum_{m=0}^{N-1} U_m(z) + U_N(z)\Lambda_N(\sigma) \right\} \tag{223}$$

where the asymptotic terms $U_m(z)$ are defined by the equation

$$U_m(z) = \frac{(\nu^2-(\frac{1}{2})^2)(\nu^2-(\frac{3}{2})^2)\cdots(\nu^2-(m-\frac{3}{2})^2)(\nu^2-(m-\frac{1}{2})^2)}{m!(2z)^m} \tag{224}$$

and $\Lambda_N(\sigma)$ is a converging factor. According to the formulae on page 547 of Reference 1, the converging factor $\Lambda_N(\sigma)$ is given in terms of the deviation

$$\sigma = \sqrt{x^2+y^2} - \tfrac{1}{2}N \tag{225}$$

and the tangent

$$\tan\theta = \frac{y}{x+\sqrt{x^2+y^2}} \tag{226}$$

by the equation

$$\Lambda_N = \frac{1}{2} + \left(\frac{\sigma}{2}+\frac{1}{8}\right)\frac{\sec^2\theta}{N} - \left[\frac{\sigma^2}{2}+\frac{\sigma}{8}+\frac{3-8\nu^2}{32}+\left(\frac{3\sigma}{8}+\frac{5}{32}\right)\tan^2\theta\right]\frac{\sec^2\theta}{N^2}$$

$$+ i\left[\frac{1}{2}-\frac{\sec^2\theta}{8N}-\left(\frac{\sigma^2}{2}+\frac{\sigma}{4}+\frac{1}{32}-\frac{3}{32}\tan^2\theta\right)\frac{\sec^2\theta}{N^2}\right]\tan\theta \tag{227}$$

With this converging factor and double precision it is possible to obtain better than single precision values for use in rational approximation.

Otherwise, summation of the asymptotic series is continued until there is no change in the sum or until $m = 35$.

After the Bessel functions of lowest order have been computed, the Bessel functions of higher order can be generated with the aid of the recurrence equation

$$K_{\nu+1}(z) - K_{\nu-1}(z) = \frac{2\nu}{z} K_\nu(z) \tag{228}$$

A rounding error ϵ_μ is introduced during the μth cycle of recurrence and persists through the subsequent cycles. The error can be expressed as a linear combination of the functions $(-1)^\nu I_\nu(z)$ and $K_\nu(z)$ both of which obey the same recurrence equations.

In view of the recurrence equation

$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z} \tag{229}$$

the persisting error in the νth cycle is given by the expression

$$z \left\{(-1)^{\nu-\mu} K_{\mu-1}(z) I_\nu(z) + I_{\mu-1}(z) K_\nu(z)\right\} \epsilon_\mu \tag{230}$$

Insofar as the function $K_\nu(z)$ increases with order faster than $I_\nu(z)$, the relative rounding error remains bounded.

Programming

SUBROUTINE CBSSLK (AZ, CN, FK)

```
******************************************************************************************
FORTRAN SUBROUTINE FOR MODIFIED BESSEL FUNCTION OF SECOND KIND
******************************************************************************************
```

The real and imaginary parts of the argument z are given in array AZ, and the order ν is given in CN. The complex Bessel function of one–third integral order is computed by series expansions, rational approximations, and recurrence relations. The real and imaginary parts of the function $K_\nu(z)$ are stored in array FK.

DISCUSSION

The subroutines for this report were programmed originally for the Naval Ordnance Research Calculator. When the NORC was about to be demolished, the subroutines were converted to a hybrid FORTRAN which would run with the STRETCH compiler. Register storage in the NORC had the same function as the accumulator in the STRETCH, but there is no way to refer to the accumulator through FORTRAN, so the register storage was simulated by a storage location in memory. This contributed to a two-fold loss of efficiency which has been observed with the hybrid FORTRAN. The subroutines have been reprogrammed from their formulations directly into FORTRAN in order to improve efficiency and accuracy.

In the present work the approximation of converging factors was accomplished in two steps, of which the first was an adjustment of the rational approximation and the second was a conversion to a polar expansion.

Future development of expansions in poles and residues probably will take the form of transformations to Stieltjes integrals which then can be approximated by Laguerre integration. Given any asymptotic expansion, the integrand of the Stieltjes integral can be so constructed, from a series of Laguerre functions, as to have the same asymptotic expansion, whence Gauss–Laguerre quadrature would lead to an approximation in poles and residues.

Another approach to the approximation of functions follows from the fact that the real and the imaginary parts of an analytic function are solutions of Laplace's equation. A sourcewise representation of the solution of Laplace's equation consists of poles with residues.

CONCLUSION

A number of functions can be approximated successfully by sets of poles with residues. The functions first were approximated as rational functions, then were converted to the equivalent singular functions.

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APPENDIX A

ALGORITHMS

METHOD OF APPROXIMATION

Rational Functions

A double precision complex routine was available on NORC for the rational approximation of a function within a half circle of unit radius. Input consisted of values of the argument z and values of a function $R(z)$ so closely spaced that interpolation gave full accuracy. The program constructed approximations

$$R(z) = \frac{p(z)}{q(z)} \tag{1}$$

of progressively higher order until a tolerance was met. Output consisted of the values of the coefficients of the polynomials $p(z)$ and $q(z)$.

In the rational function of the nth degree there were $2n+1$ complex coefficients, which were specified by the $2n+1$ complex values of the function to be approximated, at $2n+1$ nodes on the complex contour. The origin at $z=0$ and the corners at $z=\pm i$ were included among the nodes, but since there must be an odd number of nodes, the point at $z=\pm 1$ could not be a node.

The function $R(z)$ was specified to be symmetric with respect to the real axis and the polynomials $p(z)$ and $q(z)$ had real coefficients. Values of the function on the positive quarter circle were sufficient to establish the real coefficients. The method of inverted differences was used to determine the coefficients. The relative error $\epsilon(z)$ was computed for each datum of input.

The coordinates of the nodes were adjusted to make the maxima of error between nodes more uniform. The coordinate of the kth node on the imaginary axis was the distance y_k from the origin and the coordinate of the kth node on the circular arc was the angle θ_k with respect to the real axis.

Inasmuch as the error was analytic within the complex contour it could be factored into a polynomial which had roots at the nodal points on the contour. The error $\epsilon(z)$ thus could be expressed by the equation

$$\epsilon(z) = A(z) \prod_{k=1}^{k=2N+1} (z - z_k) \tag{2}$$

where $A(z)$ is an analytic function and z_k is the kth nodal point. Insofar as $A(z)$ was nearly constant, an increment in error $\Delta\epsilon$ was expressed in terms of displacements of nodes Δz_k by the equation

$$\Delta\epsilon = -\,\epsilon \sum_{k=1}^{k=2N+1} \frac{\Delta z_k}{(z - z_k)} \tag{3}$$

The increment $\Delta|\epsilon|$ in the amplitude of error $|\epsilon|$ was given by the equation

$$\Delta|\epsilon| = -\,\tfrac{1}{2}|\epsilon| \sum_{k=1}^{k=2N+1} \left[\frac{\Delta z_k}{z - z_k} + \frac{\Delta z_k^*}{z^* - z_k^*}\right] \tag{4}$$

On the imaginary axis the increment Δz_k was given in terms of distance Δy_k by the equation

$$\Delta z_k = i\Delta y_k \tag{5}$$

and on the unit circle the increment Δz_k was given in terms of angle $\Delta\theta_k$ by the equation

$$\Delta z_k = ie^{i\theta_k}\Delta\theta_k \tag{6}$$

Equalization of two successive maxima of error was expressed by the equation

$$\Delta|\epsilon|_m - \Delta|\epsilon|_{m-1} = -\{|\epsilon|_m - |\epsilon|_{m-1}\} \tag{7}$$

where $|\epsilon|_m$ was the amplitude of error at the mth maximum of error. Substitutions from Equations (4), (5), (6) in Equation (7) led to a system of simultaneous equations which could be solved by a standard method to give increments Δy_k, $\Delta\theta_k$.

After the maxima of error were uniform along the imaginary axis and were uniform along the circular arc, the number of nodes on either line was increased according to whichever line had the larger maxima of error. The coordinates on the line were interlineated to make space for the added node.

This algorithm for approximation in a finite zone within a half circle of unit radius was adapted to the problem of approximation in the infinite zone outside a circle of unit radius when the variable z was replaced by the reciprocal $\bar{z}$ in the choice of argument.

Equivalent Poles

A ratio between two power polynomials can be expressed as the sum of singular terms if the degree of the numerator is less than the degree of the denominator. The singular terms represent poles which are situated at the roots of the denominator. Let the ratio $R(z)$ be expressed by the equation

$$R(z) = \frac{p(z)}{q(z)} = \frac{a_0 + \cdots + a_{n-1}z^{n-1}}{c_0 + \cdots + c_n z^n} \tag{8}$$

where c_n is equal to unity. Let the roots of $q(z)$ be $z_1, \ldots, z_n$. It is possible to take out a term which contains only z_n, as expressed by the equation

$$R(z) = \frac{r_n}{z - z_n} + \frac{a'_0 + \cdots + a'_{n-2}z^{n-2}}{c'_0 + \cdots + c'_{n-1}z^{n-1}} \tag{9}$$

where r_n is a constant, $a'_0, \ldots, a'_{n-2}$ are the coefficients of a polynomial of lower degree, and $c'_0, \ldots, c'_{n-1}$ are the coefficients of the quotient

$$c'_0 + \cdots + c'_{n-1}z^{n-1} = \frac{c_0 + \cdots + c_n z^n}{z - z_n} \tag{10}$$

In order to evaluate the coefficients, let Equation (9) be reduced to a common denominator, whence the numerator becomes $p(z)$ and the denominator becomes $q(z)$.

When the terms of the same degree in z are equated on both sides, then the result is the following system of equations.

$$\left.\begin{aligned} a_0 &= c'_0 r_n - a'_0 z_n \\ &\cdots \\ a_k &= c'_k r_n - a'_k z_n + a'_{k-1} \\ &\cdots \\ a_{n-1} &= c'_{n-1} r_n + a'_{n-2} \end{aligned}\right\} \tag{11}$$

If these equations are multiplied in succession by the factors $1, \ldots, {z_n}^k, \ldots, {z_n}^{n-1}$ and are added, then the last two terms on the right cancel in pairs, and the terms which remain lead to the equation

$$r_n = \frac{a_0 + \cdots + a_{n-1} {z_n}^{n-1}}{c'_0 + \cdots + c'_{n-1} {z_n}^{n-1}} \tag{12}$$

Thus it is possible to evaluate r_n by a straightforward computation.

Rearrangement of the Equations (11) leads to the chain of equations

$$\left.\begin{aligned} a'_{n-2} &= a_{n-1} - c'_{n-1} r_n \\ &\cdots \\ a'_{k-1} &= a_k - c'_k r_n + a'_k z_n \\ &\cdots \\ a'_0 &= a_1 - c'_1 r_n + a'_1 z_n \end{aligned}\right\} \tag{13}$$

which can be solved successively to obtain the coefficients $a'_{n-2}, \ldots, a'_0$ of the reduced polynomial. The ratio $R(z)$ thus can be expanded recursively into the sum

$$R(z) = \frac{r_1}{z - z_1} + \cdots + \frac{r_n}{z - z_n} \tag{14}$$

where $z_1, \ldots, z_n$ are the positions of poles and $r_1, \ldots, r_n$ are the residues of the poles.

In the application of this algorithm to rational approximations, the ratio

$$\frac{p(\bar{z})}{q(\bar{z})} = \frac{a_0 + \cdots + a_n \bar{z}^n}{c_0 + \cdots + c_n \bar{z}^n} = \frac{a_n + \cdots + a_0 z^n}{c_n + \cdots + c_0 z^n} \tag{15}$$

can be resolved into the sum of the two terms

$$1 + \frac{(a_n - c_n) + \cdots + (a_1 - c_1) z^{n-1}}{c_n + \cdots + c_0 z^n} \tag{16}$$

because $a_0 \equiv c_0$, and then the second term can be expanded into singular terms. The

factor $\bar{z}$ can be included in the expansion to give the equation

$$\bar{z}\frac{p(\bar{z})}{q(\bar{z})} = \frac{1 - \dfrac{r_1}{z_1} - \cdots - \dfrac{r_n}{z_n}}{z} + \frac{\dfrac{r_1}{z_1}}{z - z_1} + \cdots + \frac{\dfrac{r_n}{z_n}}{z - z_n} \tag{17}$$

which contains the pole at the origin, while the residues are those for the expansion of the second term in Expression (16).

APPENDIX B

RATIONAL APPROXIMATIONS

TABLE I

COMPLEX EXPONENTIAL INTEGRAL

$Ei(z)$

POLES

Positions	Residues
0.000000000000000 E 00	8.501565161210931 E−03
3.111059570865283 E−02	5.050374658490585 E−02
1.036612605391116 E−01	8.368173689564071 E−02
2.165323352445536 E−01	1.070475824176067 E−01
3.699314279601916 E−01	1.204247190294619 E−01
5.667662599905892 E−01	1.250966315822293 E−01
8.140420663247483 E−01	1.223144352246853 E−01
1.123842475408128 E 00	1.126214175539071 E−01
1.514004781485123 E 00	9.634194073925819 E−02
2.008867950322836 E 00	7.473984227575106 E−02
2.640524118235915 E 00	5.085961359534413 E−02
3.450984499333923 E 00	2.908227067736279 E−02
4.495833607632020 E 00	1.322016405301009 E−02
5.850582634098224 E 00	4.438029398290675 E−03
7.622735014633804 E 00	9.926124789875764 E−04
9.978145015845783 E 00	1.265797951120111 E−04
1.321220648964083 E 01	7.021509082533505 E−06
1.803229483760214 E 01	9.102815325646319 E−08

TABLE II

COMPLEX FRESNEL INTEGRAL

$E(z)$

POLES

Positions	Residues
0.000000000000000 E 00	8.157230833240962 E−02
2.086058560134765 E−02	1.592852852534368 E−01
8.298069404956873 E−02	1.485816256144991 E−01
1.854216533260787 E−01	1.332196708362453 E−01
3.279634793823607 E−01	1.156903928789572 E−01
5.126752799128284 E−01	9.785809594475354 E−02
7.454129580451047 E−01	8.059088342976243 E−02
1.036950674182965 E 00	6.402045386098722 E−02
1.403780612554370 E 00	4.814452427678847 E−02
1.868916622140010 E 00	3.335406584732945 E−02
2.463148305239293 E 00	2.055480994701934 E−02
3.227193837373523 E 00	1.078474038875057 E−02
4.215343482800130 E 00	4.556348922142192 E−03
5.501788731515490 E 00	1.439844581389254 E−03
7.192589666831019 E 00	3.070561398341705 E−04
9.451702080764080 E 00	3.781565411685414 E−05
1.257107183147839 E 01	2.051735096161211 E−06
1.724835372163339 E 01	2.635648236827474 E−08

TABLE III

DOUBLE EXPONENTIAL INTEGRAL

$Di(z)$

POLES

Positions	Residues
0.000000000000000 E 00	−3.469117335358920 E−04
4.195566783742928 E−02	−6.037877324617450 E−03
1.175336616486651 E−01	−1.524613059492494 E−02
2.285602374559868 E−01	−2.105821698272908 E−02
3.756673501612400 E−01	−1.718942087207540 E−02
7.915948462766724 E−01	3.143234670330322 E−02
1.075468896230582 E 00	7.508985315669721 E−02
1.426592080308411 E 00	1.246897878072599 E−01
1.862905549523766 E 00	1.685790750900348 E−01
2.407300095098562 E 00	1.917150806995113 E−01
3.088540356075242 E 00	1.826007945508359 E−01
3.942776051552587 E 00	1.423456743071465 E−01
5.015931965439807 E 00	8.748622224193267 E−02
6.367591807486506 E 00	4.021750832884253 E−02
8.077761930960550 E 00	1.285750056801799 E−02
1.025989611388872 E+01	2.576737825984412 E−03
1.308967684226102 E+01	2.759550037843490 E−04
1.688321690859160 E+01	1.191393155171215 E−05
2.240832409417126 E+01	1.072929801993859 E−07

TABLE IV

DOUBLE FRESNEL INTEGRAL

$D(z)$

POLES

Positions	Residues
0.000000000000000 E 00	1.946702176629290 E−03
6.158254330058267 E−03	−5.602038332003368 E−03
5.577129558799173 E−02	−7.734209998293494 E−03
2.613294506565088 E−01	1.923681980789794 E−02
4.197557282462471 E−01	4.685483890266400 E−02
6.191129560306540 E−01	7.868054439865047 E−02
8.663690369835087 E−01	1.103380471995033 E−01
1.172669313738075 E 00	1.367629763844977 E−01
1.553763009533203 E 00	1.519362601026446 E−01
2.030480287272866 E 00	1.502574885873442 E−01
2.629550653406246 E 00	1.296972574495706 E−01
3.385002037633292 E 00	9.482725744458265 E−02
4.340433251697753 E 00	5.639680002398576 E−02
5.552727526730474 E 00	2.585885948785604 E−02
7.098490802164333 E 00	8.504554154209225 E−03
9.086373817410945 E 00	1.812509189092737 E−03
1.168418085479360 E+01	2.144626967496620 E−04
1.519234066877839 E+01	1.074839393817003 E−05
2.033584655600194 E+01	1.219304804726968 E−07

TABLE V

EXPONENTIAL EXPONENTIAL INTEGRAL

$k(z)$

POLES

Positions	Residues
0.000000000000000 E 00	3.495172589268265 E−02
2.372861283136832 E−02	1.358491059258968 E−01
8.541132106687596 E−02	1.588505815522956 E−01
1.852766272820592 E−01	1.530014345354354 E−01
3.237415266166881 E−01	1.345207528564613 E−01
5.030454603812668 E−01	1.119130516196707 E−01
7.288066075871879 E−01	8.920083866561896 E−02
1.011227701028717 E 00	6.792272054720666 E−02
1.365984481712494 E 00	4.867231978872109 E−02
1.815101390389295 E 00	3.201709765322659 E−02
2.388247019554190 E 00	1.870089650211108 E−02
3.124905320088117 E 00	9.297084144278655 E−03
4.078024898944454 E 00	3.726047631610865 E−03
5.320335455548645 E 00	1.119895375598227 E−03
6.956243072905789 E 00	2.280574968723531 E−04
9.147599025470312 E 00	2.695962277814528 E−05
1.218290743885443 E 01	1.412554302243009 E−06
1.675113119698726 E 01	1.763523268088063 E−08

TABLE VI

EXPONENTIAL FRESNEL INTEGRAL

$m(z)$

POLES

Positions	Residues
0.000000000000000 E 00	1.171632185997974 E−01
1.882000761332742 E−02	2.084876153495584 E−01
7.650297830721689 E−02	1.720712026818202 E−01
1.731094073268224 E−01	1.374456548793750 E−01
3.087247749376508 E−01	1.071559279197100 E−01
4.852407450425483 E−01	8.191163553714233 E−02
7.079315023126571 E−01	6.130560309120205 E−02
9.866948545085860 E−01	4.448272993284047 E−02
1.336964217293807 E 00	3.070308658007574 E−02
1.780586961924434 E 00	1.961788115885990 E−02
2.347109262385231 E 00	1.120676960634960 E−02
3.075935359899532 E 00	5.480694573733344 E−03
4.019969426341625 E 00	2.171717980701646 E−03
5.251927395269674 E 00	6.482424452554320 E−04
6.876223046863616 E 00	1.316220788022128 E−04
9.054707180068234 E 00	1.556852300136330 E−05
1.207577867202912 E 01	8.187688247021873 E−07
1.662778013418871 E 01	1.029295020822967 E−08

TABLE VII

BESSEL FUNCTION OF ORDER ZERO

$K_0(z)$

POLES

Positions	Residues
0.000000000000000 E 00	0.000000000000000 E 00
−1.648995051422117 E−02	−4.809423363874473 E−03
−7.186218800685365 E−02	−1.313662003477595 E−02
−1.670868781248656 E−01	−1.948438340084579 E−02
−3.025822502194688 E−01	−2.199489000320033 E−02
−4.806139452459267 E−01	−2.093966256765194 E−02
−7.070752393578979 E−01	−1.746002684586503 E−02
−9.929957905395160 E−01	−1.279378133620848 E−02
−1.355839256125922 E 00	−8.052344217965918 E−03
−1.821059078991320 E 00	−4.158173750027601 E−03
−2.424821753108787 E 00	−1.643177387479224 E−03
−3.219566557087496 E 00	−4.491755853147087 E−04
−4.286580772483836 E 00	−7.285947655740069 E−05
−5.770228167981279 E 00	−5.382652306582855 E−06
−8.013712609525260 E 00	−9.937790480362892 E−08

TABLE VIII

BESSEL FUNCTION OF ORDER ONE-THIRD

$K_{\frac{1}{3}}(z)$

POLES

Positions	Residues
0.000000000000000 E 00	0.000000000000000 E 00
−1.436346451938738 E−02	−3.308130422855452 E−03
−6.829706137621937 E−02	−7.992541489780344 E−03
−1.626145943132923 E−01	−1.114953951149366 E−02
−2.974538543947166 E−01	−1.214944767056973 E−02
−4.749052692100660 E−01	−1.131050261572697 E−02
−7.007380209350545 E−01	−9.290616383535283 E−03
−9.858961485754878 E−01	−6.738338829864222 E−03
−1.347791024780270 E 00	−4.212029053420502 E−03
−1.811850277214040 E 00	−2.165634872117979 E−03
−2.414232693109992 E 00	−8.537655388612388 E−04
−3.207380096917656 E 00	−2.332000490440423 E−04
−4.272584301588853 E 00	−3.784577154525431 E−05
−5.754199732132192 E 00	−2.800396179763318 E−06
−7.995346895730257 E 00	−5.183943839708712 E−08

TABLE IX

BESSEL FUNCTION OF ORDER TWO–THIRDS

$K_{\frac{2}{3}}(z)$

POLES

Positions	Residues
0.000000000000000 E 00	0.000000000000000 E 00
−1.002408161647787 E−02	8.356269613036070 E−03
−5.997870547395276 E−02	1.421589792080583 E−02
−1.516343542993466 E−01	1.661492960434507 E−02
−2.845368398965238 E−01	1.632135891550809 E−02
−4.603218236550183 E−01	1.421532926606566 E−02
−6.844133505234451 E−01	1.116165182456759 E−02
−9.675114349534103 E−01	7.846521235416155 E−03
−1.326876006320234 E 00	4.800776505229076 E−03
−1.787857730177285 E 00	2.433841308314365 E−03
−2.386589640411052 E 00	9.515251496981341 E−04
−3.175517091435769 E 00	2.589249109738377 E−04
−4.235942093386486 E 00	4.201905951784872 E−05
−5.712193661037926 E 00	3.118825106795326 E−06
−7.947172476979304 E 00	5.808362400385792 E−08

TABLE X

BESSEL FUNCTION OF ORDER ONE

$K_1(z)$

POLES

Positions	Residues
0.000000000000000 E 00	0.000000000000000 E 00
−5.577424298795054 E−03	7.538057792005914 E−02
−4.991129441724757 E−02	7.122935374034643 E−02
−1.374409116523968 E−01	6.331162242281997 E−02
−2.672337847105657 E−01	5.282402645233011 E−02
−4.403801668086817 E−01	4.133053594414916 E−02
−6.618136148725406 E−01	3.013505739475096 E−02
−9.418610776650166 E−01	2.010434395927201 E−02
−1.297541304683261 E 00	1.185522230680744 E−02
−1.754076967198164 E 00	5.860555109560099 E−03
−2.347552998822763 E 00	2.254651482673253 E−03
−3.130413326891964 E 00	6.081730415363355 E−04
−4.183971205637291 E 00	9.842155506257467 E−05
−5.652517992149936 E 00	7.321390930380890 E−06
−7.878639598106769 E 00	1.372796673846658 E−07

APPENDIX C

ERRORS OF APPROXIMATION

MAXIMUM RELATIVE ERROR

FUNCTION	ERROR	
	Axis	Arc
$Ei(z)$	11.6×10^{-16}	4.4×10^{-16}
$E(z)$	4.1×10^{-16}	1.8×10^{-16}
$Di(z)$	5.1×10^{-16}	2.2×10^{-16}
$D(z)$	4.8×10^{-16}	2.0×10^{-16}
$k(z)$	2.8×10^{-16}	1.4×10^{-16}
$m(z)$	1.7×10^{-16}	0.9×10^{-16}
$K_0(z)$	3.1×10^{-16}	1.3×10^{-16}
$K_{\frac{1}{3}}(z)$	1.7×10^{-16}	0.7×10^{-16}
$K_{\frac{2}{3}}(z)$	2.0×10^{-16}	0.8×10^{-16}
$K_1(z)$	5.1×10^{-16}	2.1×10^{-16}

APPENDIX D

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13. ABSTRACT

A few special functions have been approximated in the complex plane with rational functions. The error bounds of the approximations conform to the Chebyshev criterion. The rational functions have been converted into equivalent expansions in terms of the singular functions for poles and residues. Analyses and programs are described for ten functions.